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# DIOPHANTINE APPROXIMATION WITH FOUR SQUARES AND ONE *K*-TH POWER OF PRIMES

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## Abstract

We show that if $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \mu$ are non-zero real numbers, not all of the same sign, $\nu$ is real, and at least one of the ratios $\lambda_i / \lambda_j$ is irrational, then for $0 < \sigma < \frac{1}{3k2^k}$ and any positive integer $k \geq 3$, the inequality $|\lambda_1 p_1^2 + \lambda_2 p_2^2 + \lambda_3 p_3^2 + \lambda_4 p_4^2 + \mu p_5^k + \nu| < (\max p_j)^{-\sigma}$, has infinitely many primes solutions $(p_1, \cdots, p_5)$.

## 1. Introduction

In 1946, Davenport and Heilbronn [3] proved that if $\lambda_1, \cdots, \lambda_5$ are non-zero real numbers, not all of the same sign, with one at least of the

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ratios $\lambda_i / \lambda_j$ irrational, then for any $\varepsilon > 0$, there exist integers $x_1, \cdots, x_5$, not all zero, such that

$$|\lambda_1 x_1^2 + \cdots + \lambda_5 x_5^2| < \varepsilon. \tag{1.1}$$

Later, Birch and Davenport [2] showed that a solution of (1.1) exists with $x_1, \cdots, x_5$ all of order $\varepsilon^{-2-\delta}$ for any $\delta > 0$, thus there exist infinitely many sets of integers $x_1, \cdots, x_5$ satisfying the inequality

$$|\lambda_1 x_1^2 + \cdots + \lambda_5 x_5^2| < \max|x_i|^{-\frac{1}{2}+\delta}.$$

Further, Bambah [1] extended Davenport and Heilbronn's result [3] and proved if $\lambda_1, \cdots, \lambda_4, \mu$ are non-zero real numbers, not all of the same sign, and at least one ratio $\lambda_i / \lambda_j$ is irrational, then for any positive integer $k$, any real number $\nu$ and any $\varepsilon > 0$, the inequality

$$|\lambda_1 x_1^2 + \cdots + \lambda_4 x_4^2 + \mu x_5^k + \nu| < \varepsilon, \tag{1.2}$$

has infinitely many solutions in positive integers $x_1, \cdots, x_5$.

In 1978, Lau and Liu [6] gave a quantitative form of Bambah's result.

Our purpose here is to investigate that, under some necessary conditions, whether the result of Bambah [1] is true when integers $x_j$ in (1.2) are replaced by primes $p_j$.

The result we obtain is as follows:

**Theorem.** *Suppose that* $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \mu$ *are non-zero real numbers, not all of the same sign, that* $\nu$ *is real, and that at least one of the ratios* $\lambda_i / \lambda_j$ *is irrational, then for* $0 < \sigma < \dfrac{1}{3k2^k}$ *and any positive integer* $k \geq 3$, *the inequality*

$$|\lambda_1 p_1^2 + \lambda_2 p_2^2 + \lambda_3 p_3^2 + \lambda_4 p_4^2 + \mu p_5^k + \nu| < (\max p_j)^{-\sigma}, \tag{1.3}$$

*has infinitely many primes solutions* $(p_1, \cdots, p_5)$.

We note the reason we let integer $k \geq 3$ in our theorem is that for the case $k = 1$, (1.3) is trivial since the authors [7] have obtained the analogue result about diophantine approximation with one prime and three squares of primes, and for the case $k = 2$, one may see the results of Vaughan [9] and Harman [5].

## 2. Notation

Throughout, integer $k \geq 3$, $\delta$, and $\eta$ are small positive numbers. Constants both explicit and implicit, in the $O$, $\ll$ and $\gg$ notations, shall depend only on $\lambda_1, \cdots, \lambda_4$, $\mu$, and $\nu$.

Without loss of generality, we may assume that $\lambda_1 / \lambda_2$ is irrational, then there are infinitely many pairs of integers $q$, $a$ with $|\lambda_1 / \lambda_2 - a / q| \leq q^{-2}$, $(a, q) = 1$, $q > 0$, and $a \neq 0$. We choose $q$ to be large in terms of $\lambda_1, \cdots, \lambda_4$, $\mu$, and $\nu$. We write $e(x) = \exp(2\pi i x)$ and make the following definitions.

$$X^{1-\frac{1}{3k}} L^{-1} = q, \quad L = \log X, \quad T^2 = T'^k = X^{\frac{1}{3}}, \quad \tau = X^{-1+\frac{1}{6k}}, \tag{2.1}$$

$$Q = (|\lambda_1|^{-1} + |\lambda_2|^{-1}) X^{1-\frac{1}{6k}}, \quad \varepsilon = X^{-\omega}, \quad 0 < \omega < \frac{1}{6k2^k}, \tag{2.2}$$

$$g_j(x) = \sum_{\delta X^{\frac{1}{2}} \leq p \leq X^{\frac{1}{2}}} e(\lambda_j p^2 x) \log p, \quad j = 1, 2, 3, 4,$$

$$h(x) = \sum_{\eta X^{\frac{1}{k}} \leq p \leq X^{\frac{1}{k}}} e(\mu p^k x) \log p.$$

Let further

$$K_\varepsilon(x) = \left( \frac{\sin \pi \varepsilon x}{\pi x} \right)^2, \ (x \neq 0), \quad K_\varepsilon(0) = \varepsilon^2,$$

then

$$K_\varepsilon(x) \ll \min(\varepsilon^2, x^{-2}). \tag{2.3}$$

## 3. Outline of the Method

Our method goes back to Vaughan [8], [9], and the basis of the Davenport-Heilbronn adaption of the circle method is the identity

$$\int_{-\infty}^{\infty} e(xy)K_{\varepsilon}(x)dx = \max(0, \varepsilon - |y|), \tag{3.1}$$

which is a trivial corollary of Lemma 4 of Davenport and Heilbronn [3].

**Lemma 3.1.** *Let* $\rho = \beta + i\gamma$ *be the zeros of the Riemann Zeta function. Suppose that* $2 \le Y \le X$, $T \ge 2$, *then*

$$\sum_{p \le Y} \log p = Y - \sum_{|\gamma| \le T, \beta \ge \frac{2}{3}} Y^{\rho}\rho^{-1} + O\left(YT^{-1}(\log YT)^2 + Y^{\frac{2}{3}}(\log T)^2\right).$$

**Proof.** See [8]. □

We note that

$$\int_{\eta X^{\frac{1}{k}}}^{X^{\frac{1}{k}}} e(\mu x y^k)dy = \int_{\eta^k X}^{X} \frac{1}{k} y^{\frac{1}{k}-1} e(\mu x y)dy \ll X^{\frac{1}{k}-1} \min(X, |x|^{-1})$$

$$\ll \min\left(X^{\frac{1}{k}}, |x|^{-\frac{1}{k}}\right).$$

Using Lemma 3.1 and partial integration, we have

$$h(x) = \int_{\eta X^{\frac{1}{k}}}^{X^{\frac{1}{k}}} e(\mu x Y^k)d\left(Y - \sum_{|\gamma| \le T', \beta \ge \frac{2}{3}} \frac{Y^{\rho}}{\rho} + O\left(YT'^{-1}(\log YT')^2 + Y^{\frac{2}{3}}(\log T')^2\right)\right)$$

$$=: \int_{\eta X^{\frac{1}{k}}}^{X^{\frac{1}{k}}} e(\mu x Y^k)dY - \int_{\eta X^{\frac{1}{k}}}^{X^{\frac{1}{k}}} e(\mu x Y^k)d\left(\sum_{|\gamma| \le T', \beta \ge \frac{2}{3}} \frac{Y^{\rho}}{\rho}\right)$$

$$+\int_{\eta X^{\frac{1}{k}}}^{X^{\frac{1}{k}}} e(\mu x Y^k)\, d(O(R(Y, T')))$$

$$= \int_{\eta X^{\frac{1}{k}}}^{X^{\frac{1}{k}}} e(\mu x y^k)\, dy - \sum_{|\gamma|\le T', \beta\ge\frac{2}{3}} \sum_{\eta X^{\frac{1}{k}} \le n \le X^{\frac{1}{k}}} n^{\rho-1} e(\mu x n^k)$$

$$+ O\left(\left(1 + |x| X^{\frac{k-1}{k}} \left| \int_{\eta X^{\frac{1}{k}}}^{X^{\frac{1}{k}}} e(\mu x y^k)\, dy \right| \right) X^{\frac{2}{3k}} L^2 \right)$$

$$= \int_{\eta X^{\frac{1}{k}}}^{X^{\frac{1}{k}}} e(\mu x y^k)\, dy - \sum_{|\gamma|\le T', \beta\ge\frac{2}{3}} \sum_{\eta X^{\frac{1}{k}} \le n \le X^{\frac{1}{k}}} n^{\rho-1} e(\mu x n^k)$$

$$+ O\left(\left(1 + |x|^{1-\frac{1}{k}} X^{1-\frac{1}{k}}\right) X^{\frac{2}{3k}} L^2 \right)$$

$$=: I_5(x) - J_5(x) + B_5(x). \tag{3.2}$$

And so

$$g_j(x) = \int_{\delta X^{\frac{1}{2}}}^{X^{\frac{1}{2}}} e(\lambda_j x y^2)\, dy - \sum_{|\gamma|\le T, \beta\ge\frac{2}{3}} \sum_{\delta X^{\frac{1}{2}} \le n \le X^{\frac{1}{2}}} n^{\rho-1} e(\lambda_j x n^2)$$

$$+ O\left(\left(1 + |x|^{\frac{1}{2}} X^{\frac{1}{2}}\right) X^{\frac{1}{3}} L^2 \right)$$

$$=: I_j(x) - J_j(x) + B_j(x), \quad j = 1, 2, 3, 4. \tag{3.3}$$

On interchanging the order of summation and integration, and by (3.1), we have

$$\int_{-\infty}^{\infty} \prod_{j=1}^{4} g_j(x) h(x) e(x\nu) K_\varepsilon(x)\, dx$$

$$= \sum_{\substack{\delta X^{\frac{1}{2}} \le p_1, p_2, p_3, p_4 \le X^{\frac{1}{2}} \\ \eta X^{\frac{1}{k}} \le p_5 \le X^{\frac{1}{k}}}} \prod_{j=1}^{5} \log p_j \int_{-\infty}^{\infty} e\left(\left(\sum_{j=1}^{4} \lambda_j p_j^2 + \mu p_5^k + \nu\right) x\right) K_\varepsilon(x) dx$$

$$= \sum_{\substack{\delta X^{\frac{1}{2}} \le p_1, p_2, p_3, p_4 \le X^{\frac{1}{2}} \\ \eta X^{\frac{1}{k}} \le p_5 \le X^{\frac{1}{k}}}} \prod_{j=1}^{5} \log p_j \max\left(0, \varepsilon - \left|\sum_{j=1}^{4} \lambda_j p_j^2 + \mu p_5^k + \nu\right|\right)$$

$$\le \frac{1}{16k} \varepsilon L^5 \sum_{\substack{\delta X^{\frac{1}{2}} \le p_1, p_2, p_3, p_4 \le X^{\frac{1}{2}}, \eta X^{\frac{1}{k}} \le p_5 \le X^{\frac{1}{k}} \\ |\lambda_1 p_1^2 + \lambda_2 p_2^2 + \lambda_3 p_3^2 + \lambda_4 p_4^2 + \mu p_5^k + \nu| < \varepsilon}} 1$$

$$=: \frac{1}{16k} \varepsilon L^5 J(X),$$

hence

$$J(X) \ge 16k\varepsilon^{-1} L^{-5} \int_{-\infty}^{\infty} \prod_{j=1}^{4} g_j(x) h(x) e(x\nu) K_\varepsilon(x) dx. \tag{3.4}$$

Next, we shall prove that the integral in (3.4) is $\varepsilon^2 X^{1+\frac{1}{k}}$ in order of magnitude. As usual, we split the range of infinite integration into three sections

$$\mathfrak{M} = \{x : |x| \le \tau\}, \ \mathfrak{m} = \{x : \tau < |x| \le X^{\frac{1}{6k}}\}, \ \mathfrak{t} = \{x : |x| > X^{\frac{1}{6k}}\},$$

which are traditionally named the neighborhood of the origin, the intermediate region, and the trivial region. The dominant contribution to this infinite integral is from the neighborhood of the origin, and there $\prod_{j=1}^{5} I_j(x)$ is a good approximation to $\prod_{j=1}^{4} g_j(x) h(x)$. The contribution from the trivial region is negligible. To deal with the contribution from the

intermediate region, we use the fact that $\lambda_1 / \lambda_2$ is irrational, which enables us to show one of $|g_1(x)|$ and $|g_2(x)|$ must be relatively small in the intermediate region.

## 4. The Neighborhood of the Origin

This part of the integral forms the main positive contribution to the integral.

By Lemma 8 of Vaughan [9], we get Lemmas 4.1 and 4.2.

**Lemma 4.1.** *For* $j = 1, 2, 3, 4,$ *we have*

$$g_j(x) \ll X^{\frac{1}{2}}, \quad I_j(x) \ll X^{\frac{1}{2}} \min(1, X^{-1}|x|^{-1}),$$

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} |I_j(x)|^2 dx \ll 1, \quad \int_{-\frac{1}{2}}^{\frac{1}{2}} |J_j(x)|^2 dx \ll \exp\left(-2^{\frac{4}{5}} L^{\frac{1}{5}}\right),$$

$$\int_{-\tau}^{\tau} |B_j(x)|^2 dx \ll \exp\left(-2^{\frac{4}{5}} L^{\frac{1}{5}}\right), \quad \int_{-\tau}^{\tau} |g_j(x)|^2 dx \ll 1.$$

**Lemma 4.2.** *We have*

$$h(x) \ll X^{\frac{1}{k}}, \quad I_5(x) \ll X^{\frac{1}{k}} \min(1, X^{-1}|x|^{-1}),$$

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} |I_5(x)|^2 dx \ll X^{\frac{2}{k}-1}, \quad \int_{-\tau}^{\tau} |B_5(x)|^2 dx \ll X^{\frac{2}{k}-1} \exp\left(-2k^{-\frac{1}{5}} L^{\frac{1}{5}}\right),$$

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} |J_5(x)|^2 dx \ll X^{\frac{2}{k}-1} \exp\left(-2k^{-\frac{1}{5}} L^{\frac{1}{5}}\right), \quad \int_{-\tau}^{\tau} |h(x)|^2 dx \ll X^{\frac{2}{k}-1}.$$

**Lemma 4.3.** *We have*

$$\int_{\mathfrak{M}} \left| \prod_{j=1}^{4} g_j(x) h(x) - \prod_{j=1}^{5} I_j(x) \right| K_\varepsilon(x) dx \ll \varepsilon^2 X^{1+\frac{1}{k}} L^{-1}.$$

**Proof.** By (3.2), (3.3), (2.3), Lemmas 4.1 and 4.2, we have

$$\int_{\mathfrak{M}}\left|\prod_{j=1}^{4} g_j(x)h(x)-\prod_{j=1}^{5} I_j(x)\right|K_\varepsilon(x)dx$$

$$=\int_{\mathfrak{M}}\left|\sum_{j=1}^{5}\left(\prod_{i<j} g_i(x)\right)(B_j(x)-J_j(x))\prod_{i>j} I_i(x)\right|K_\varepsilon(x)dx$$

$$\ll \varepsilon^2 X^{1+\frac{1}{k}}\int_{\mathfrak{M}}(|B_1(x)|+|J_1(x)|)|I_2(x)|dx$$

$$+\varepsilon^2 X^{1+\frac{1}{k}}\int_{\mathfrak{M}}(|B_2(x)|+|J_2(x)|)|I_3(x)|dx$$

$$+\varepsilon^2 X^{1+\frac{1}{k}}\int_{\mathfrak{M}}(|B_3(x)|+|J_3(x)|)|I_4(x)|dx$$

$$+\varepsilon^2 X^{1+\frac{1}{k}}\int_{\mathfrak{M}}|g_3(x)|(|B_4(x)|+|J_4(x)|)dx$$

$$+\varepsilon^2 X^{\frac{3}{2}}\int_{\mathfrak{M}}|g_4(x)|(|B_5(x)|+|J_5(x)|)dx. \tag{4.1}$$

Using Schwarz's inequality, we get

$$\int_{\mathfrak{M}}(|B_1(x)|+|J_1(x)|)|I_2(x)|dx$$

$$\leq\left(\int_{\mathfrak{M}}(|B_1(x)|+|J_1(x)|)^2dx\right)^{\frac{1}{2}}\left(\int_{\mathfrak{M}}|I_2(x)|^2dx\right)^{\frac{1}{2}}$$

$$\ll\exp\left(-2^{-\frac{1}{5}}L^{\frac{1}{5}}\right). \tag{4.2}$$

Similarly,

$$\int_{\mathfrak{M}}(|B_2(x)|+|J_2(x)|)|I_3(x)|dx\ll\exp\left(-2^{-\frac{1}{5}}L^{\frac{1}{5}}\right), \tag{4.3}$$

$$\int_{\mathfrak{M}} (|B_3(x)| + |J_3(x)|)|I_4(x)|dx \ll \exp\left(-2^{-\frac{1}{5}} L^{\frac{1}{5}}\right), \tag{4.4}$$

$$\int_{\mathfrak{M}} |g_3(x)|(|B_4(x)| + |J_4(x)|)dx \ll \exp\left(-2^{-\frac{1}{5}} L^{\frac{1}{5}}\right), \tag{4.5}$$

$$\int_{\mathfrak{M}} |g_4(x)|(|B_5(x)| + |J_5(x)|)dx \ll X^{\frac{1}{k}-\frac{1}{2}} \exp\left(-k^{-\frac{1}{5}} L^{\frac{1}{5}}\right). \tag{4.6}$$

Then, the desired bound now follows from (4.1)-(4.6). □

**Lemma 4.4.** *We have*

$$\int_{|x|>\tau} \left|\prod_{j=1}^{5} I_j(x)\right| K_\varepsilon(x)dx \ll \varepsilon^2 X^{1+\frac{1}{3k}},$$

*and*

$$\int_{-\infty}^{\infty} \prod_{j=1}^{5} I_j(x) e(x\nu) K_\varepsilon(x)dx \gg \varepsilon^2 X^{1+\frac{1}{k}}.$$

**Proof.** By Lemmas 4.1 and 4.2, we have

$$\int_{|x|>\tau} \left|\prod_{j=1}^{5} I_j(x)\right| K_\varepsilon(x)dx \ll \varepsilon^2 \int_{\tau}^{+\infty} X^{\frac{1}{k}-3} x^{-5} dx \ll \varepsilon^2 X^{1+\frac{1}{3k}}.$$

From (3.1), we get

$$\begin{aligned}
&\int_{-\infty}^{+\infty} \prod_{j=1}^{5} I_j(x) e(x\nu) K_\varepsilon(x)dx \\
&= \int_{\delta X^{\frac{1}{2}}}^{X^{\frac{1}{2}}} \cdots \int_{\delta X^{\frac{1}{2}}}^{X^{\frac{1}{2}}} \int_{\eta X^{\frac{1}{k}}}^{X^{\frac{1}{k}}} \max\left(0,\ \varepsilon - \left|\sum_{j=1}^{4} \lambda_j y_j^2 + \mu y_5^k + \nu\right|\right) dy_5 \cdots dy_1 \\
&= \frac{1}{16k} \int_{\delta^2 X}^{X} \cdots \int_{\delta^2 X}^{X} \int_{\eta^k X}^{X} \prod_{j=1}^{4} y_j^{-\frac{1}{2}} y_5^{\frac{1}{k}-1} \max\left(0,\ \varepsilon - \left|\sum_{j=1}^{4} \lambda_j y_j + \mu y_5 + \nu\right|\right) \\
&\qquad\qquad \times dy_5 \cdots dy_1.
\end{aligned} \tag{4.7}$$

Let $|\sum_{j=1}^{4}\lambda_j y_j + \mu y_5 + \nu| \le \frac{1}{2}\varepsilon$, and since $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \mu$ are not all the same sign, we may assume that $\lambda_1 > 0, \lambda_2, \lambda_3, \lambda_4, \mu < 0$, and the other cases can be dealt with by the same method, then

$$-\frac{1}{2}\varepsilon\lambda_1^{-1} - \left(\sum_{j=2}^{4}\lambda_j y_j + \mu y_5 + \nu\right)\lambda_1^{-1} \le y_1$$

$$\le \frac{1}{2}\varepsilon\lambda_1^{-1} - \left(\sum_{j=2}^{4}\lambda_j y_j + \mu y_5 + \nu\right)\lambda_1^{-1}.$$

Based on

$$-\frac{1}{2}\varepsilon\lambda_1^{-1} - \left(\sum_{j=2}^{4}\lambda_j y_j + \mu y_5 + \nu\right)\lambda_1^{-1} > \delta^2 X,$$

$$\frac{1}{2}\varepsilon\lambda_1^{-1} - \left(\sum_{j=2}^{4}\lambda_j y_j + \mu y_5 + \nu\right)\lambda_1^{-1} < X,$$

we take

$$\max(\delta^2, \eta^k)X < 8^{-1}\lambda_1 X\left(\sum_{j=1}^{4}|\lambda_j| + |\mu|\right)^{-1}$$

$$\le y_2, \cdots, y_5 \le 4^{-1}\lambda_1 X\left(\sum_{j=1}^{4}|\lambda_j| + |\mu|\right)^{-1},$$

therefore, (4.7)

$$\ge \frac{1}{16k}\cdot\frac{1}{2}\varepsilon X^{\frac{1}{k}-3}\varepsilon\lambda_1^{-1}\left(8\left(\sum_{j=1}^{4}|\lambda_j| + |\mu|\right)\right)^{-4}(\lambda_1 X)^4 \gg \varepsilon^2 X^{1+\frac{1}{k}},$$

which implies the claim. □

## 5. The Intermediate Region

The following Lemma 5.1 plays an important role in the treatment of the intermediate region.

**Lemma 5.1.** *Suppose that* $\alpha$ *has a rational approximation* $b/r$ *with* $(b, r) = 1$ *and* $|\alpha - \frac{b}{r}| < \frac{1}{r^2}$, *then given any real number* $\omega_1 > 0$,

$$\sum_{\delta X^{\frac{1}{2}} \le n \le X^{\frac{1}{2}}} \Lambda(n) e(\alpha n^2) \ll X^{\frac{1}{2}+\omega_1} \left( r^{-1} + X^{-\frac{1}{4}} + rX^{-1} \right)^{\frac{1}{4}},$$

*where the implicit constant depends only on* $\omega_1$.

**Proof.** This is actually Theorem 2 of Ghosh [4]. □

**Lemma 5.2.** *For every real number* $x \in \mathfrak{m}$, *let* $V(x) = \min(|S_1(x)|, |S_2(x)|)$, *then*

$$V(x) \ll X^{\frac{1}{2} - \frac{1}{24k} + \omega_1}.$$

**Proof.** For $j = 1, 2$, choose $a_j, q_j$ such that

$$|\lambda_j x - a_j / q_j| \le (q_j Q)^{-1}, \tag{5.1}$$

with $(a_j, q_j) = 1$ and $1 \le q_j \le Q$.

By (2.1) and (2.2), $\tau > |\lambda_j|^{-1} Q^{-1}$, hence $a_1 a_2 \ne 0$. If $q_1, q_2 \le X^{\frac{1}{6k}}$, then

$$\left| a_2 q_1 \frac{\lambda_1}{\lambda_2} - a_1 q_2 \right| \le \left| \frac{a_2 / q_2}{\lambda_2 x} q_1 q_2 \left( \lambda_1 x - \frac{a_1}{q_1} \right) \right|$$

$$+ \left| \frac{a_1 / q_1}{\lambda_2 x} q_1 q_2 \left( \lambda_2 x - \frac{a_2}{q_2} \right) \right| \ll X^{\frac{1}{6k}} Q^{-1} < \frac{1}{2q}.$$

We recall that $q$ was chosen as the denominator of a convergent to the continued fraction for $\lambda_1/\lambda_2$. Thus, by Legendre's law of best approximation, we have

$$\left|q'\frac{\lambda_1}{\lambda_2}-a'\right|>\frac{1}{2q},$$

for all integers $a', q'$ with $1 \le q' < q$, therefore $|a_2 q_1| \ge q$. However, by (5.1),

$$|a_2 q_1| < |\lambda_2| X^{\frac{1}{6k}} q_2 q_1 \ll X^{\frac{1}{2k}},$$

this is a contradiction, thus, we establish that for at least one $j$, $X^{\frac{1}{6k}} < q_j \ll Q$. By Lemma 5.1 and with $\alpha = \lambda_j x$, $b = a_j$, $r = q_j$ give the desired inequality. □

**Lemma 5.3.** *We have*

$$\int_{\mathfrak{m}}\left|\prod_{j=1}^{4} g_j(x)h(x)\right|K_\varepsilon(x)dx \ll \varepsilon^2 X^{1+\frac{1}{k}} L^{-1}.$$

**Proof.** By (2.3) and Hua's inequality, we have

$$\begin{aligned}
&\int_{-\infty}^{\infty}|g_j(x)|^4 K_\varepsilon(x)dx \\
&\quad\ll \sum_{n=-\infty}^{\infty}\int_n^{n+1}|g_j(x)|^4 K_\varepsilon(x)dx \\
&\quad\ll \varepsilon^2\sum_{n=0}^{[\varepsilon^{-1}]}\int_n^{n+1}|g_j(x)|^4 dx + \sum_{n=[\varepsilon^{-1}]+1}^{\infty} n^{-2}\int_n^{n+1}|g_j(x)|^4 dx \\
&\quad\ll \varepsilon X L^2, \qquad (5.2)
\end{aligned}$$

$$\int_{-\infty}^{+\infty}|h(x)|^{2^k} K_\varepsilon(x)dx$$

$$\ll \sum_{n=-\infty}^{+\infty} \int_n^{n+1} |h(x)|^{2^k} K_\varepsilon(x)dx$$

$$\ll \varepsilon^2 \sum_{n=0}^{[\varepsilon^{-1}]} \int_n^{n+1} |h(x)|^{2^k} dx + \sum_{n=[\varepsilon^{-1}]+1}^{+\infty} n^{-2} \int_n^{n+1} |h(x)|^{2^k} dx$$

$$\ll \varepsilon X^{\frac{2^k - k + \omega_2}{k}}, \tag{5.3}$$

where $\omega_2$ is a given very small positive number.

Define

$$V = \max_{x \in \mathfrak{m}} V(x),$$

by (5.2), (5.3), and Hölder's inequality, we have

$$\int_{\mathfrak{m}} \left| \prod_{j=1}^{4} g_j(x) h(x) \right| K_\varepsilon(x)dx$$

$$\leq V^{\frac{4}{2^k}} \int_{\mathfrak{m}} |g_1(x)|^{1-\frac{4}{2^k}} |g_2(x)||g_3(x)||g_4(x)||h(x)| K_\varepsilon(x)dx$$

$$+ V^{\frac{4}{2^k}} \int_{\mathfrak{m}} |g_1(x)||g_2(x)|^{1-\frac{4}{2^k}} |g_3(x)||g_4(x)||h(x)| K_\varepsilon(x)dx$$

$$\leq V^{\frac{4}{2^k}} \left( \int_{\mathfrak{m}} |g_1(x)|^4 K_\varepsilon(x)dx \right)^{\frac{1}{4} - \frac{1}{2^k}} \prod_{j=2}^{4} \left( \int_{\mathfrak{m}} |g_j(x)|^4 K_\varepsilon(x)dx \right)^{\frac{1}{4}}$$

$$\cdot \left( \int_{\mathfrak{m}} |h(x)|^{2^k} K_\varepsilon(x)dx \right)^{\frac{1}{2^k}} + V^{\frac{4}{2^k}} \left( \int_{\mathfrak{m}} |g_2(x)|^4 K_\varepsilon(x)dx \right)^{\frac{1}{4} - \frac{1}{2^k}}$$

$$\cdot \prod_{j=1,\,3,\,4} \left( \int_{\mathfrak{m}} |g_j(x)|^4 K_\varepsilon(x)dx \right)^{\frac{1}{4}} \left( \int_{\mathfrak{m}} |h(x)|^{2^k} K_\varepsilon(x)dx \right)^{\frac{1}{2^k}}$$

$$\ll \left(X^{\frac{1}{2}-\frac{1}{24k}+\omega_1}\right)^{\frac{4}{2^k}} (\varepsilon X L^2)^{1-\frac{1}{2^k}} \left(\varepsilon X^{\frac{2^k-k+\omega_2}{k}}\right)^{\frac{1}{2^k}}$$

$$\ll \varepsilon^2 X^{1+\frac{1}{k}} L^{-1}.$$

This proves the lemma. □

## 6. The Trivial Region

**Lemma 6.1.** *We have*

$$\int_{\mathfrak{t}} \left|\prod_{j=1}^{4} g_j(x) h(x)\right| K_\varepsilon(x)\,dx \ll \varepsilon^2 X^{1+\frac{1}{k}} L^{-1}.$$

**Proof.** By Schwarz's inequality and Parseval's identity, we have

$$\int_{\mathfrak{t}} \left|\prod_{j=1}^{4} g_j(x) h(x)\right| K_\varepsilon(x)\,dx$$

$$\ll X^{\frac{1}{k}} \prod_{j=1}^{4} \left(\int_{\mathfrak{t}} |g_j(x)|^4 K_\varepsilon(x)\,dx\right)^{\frac{1}{4}}$$

$$\ll X^{\frac{1}{k}} \prod_{j=1}^{4} \left(\int_{|\lambda_j| X^{\frac{1}{6k}}}^{+\infty} \left|\sum_{p^2 \le X} e(p^2 y) \log p\right|^4 y^{-2}\,dy\right)^{\frac{1}{4}}$$

$$\ll X^{\frac{1}{k}} \prod_{j=1}^{4} \left(\sum_{n > |\lambda_j| X^{\frac{1}{6k}}} n^{-2} \int_n^{n+1} \left|\sum_{p^2 \le X} e(p^2 y) \log p\right|^4 dy\right)^{\frac{1}{4}}$$

$$\ll X^{\frac{1}{k}} \prod_{j=1}^{4} \left(\sum_{n > |\lambda_j| X^{\frac{1}{6k}}} n^{-2} X L^2\right)^{\frac{1}{4}}$$

$$\ll \varepsilon^2 X^{1+\frac{1}{k}} L^{-1}.$$

□

## 7. Proof of Theorem

Lemmas 4.3, 4.4, 5.3, and 6.1 together imply that

$$\int_{-\infty}^{\infty} \prod_{j=1}^{4} g_j(x) h(x) e(x\nu) K_\varepsilon(x) dx \gg \varepsilon^2 X^{1+\frac{1}{k}},$$

so by (3.4), we know $J(X) \gg \varepsilon X^{1+\frac{1}{k}} L^{-5}$, i.e., there are $\varepsilon X^{1+\frac{1}{k}} L^{-5}$ ordered primes $(p_1, \cdots, p_5)$ with $\delta X^{\frac{1}{2}} \leq p_1, p_2, p_3, p_4 \leq X^{\frac{1}{2}}$, $\eta X^{\frac{1}{k}} \leq p_5 \leq X^{\frac{1}{k}}$, and $|\lambda_1 p_1^2 + \lambda_2 p_2^2 + \lambda_3 p_3^2 + \lambda_4 p_4^2 + \mu p_5^k + \nu| < \varepsilon$. From (2.1) and (2.2), we know $\varepsilon \leq (\max p_j)^{-\sigma}$, and $\varepsilon X^{1+\frac{1}{k}} L^{-5}$ is large ensure that (1.3) occurs for infinity many ordered primes $(p_1, \cdots, p_5)$.

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